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The two-level pairing model in the Schwinger representation

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Abstract

The two-level pairing model was investigated in the framework of the Schwinger representation. It has been shown that zero-point fluctuations, which are important in the crossover region, are well described by certain coherent states which conserve the constants of motion. These so-called conserving approximations provide excellent descriptions of the ground-state energy and of the first excitation energy over the whole range of coupling constant values. The description of the ground-state energy provided by the Glauber coherent state is also reasonable, but not so good. Although the Glauber coherent state fails to provide the description of the excitation energy around the critical point, its performance in the strong coupling limit is very good both for the description of the ground-state energy and of the first excitation energy. The present results also show that already at the mean-field level, the bosonic description in the independent particle approximation incorporates important correlation corrections to the conventional mean-field description based on the fermionic realization of the $su(2)$ algebra (BCS theory).

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1. Introduction

Correlations play an important role in quantum many-body systems. Their study is a major challenge in various fields of physics. The existence of correlations and the emergence of phase transitions are interdependent. The study of this interdependency in simple models, which has been considered by many authors [1–3], is very instructive.

In particular, the transition to superconducting or superfluid phase in interacting Fermi systems has been attracting the attention of physicists since Kamerlingh Onnes made his spectacular discovery in 1911. Important aspects of the underlying physics of this phenomenon are well described by the two-level pairing model, which has been extensively investigated by various researchers [4, 5]. It is well known that the building blocks of the two-level pairing model are Cooper-pair operators $P_i^\dagger$, P_i and fermion number operators N_i associated with each level i, constituting the generators of a $su(2) \otimes su(2)$ algebra. The model refers to a system of nucleons subject to an average potential and interacting via a pure pairing force, i.e. nucleon pairs couple to total spin zero so that their wavefunctions have maximal spacial overlap. Each level has the same degeneracy 2Ω. The general form of the Hamiltonian is

$$H = \epsilon(N_1 - N_2) + g \sum_{i,j=1}^{2} P_i^+ P_j,$$

where ϵ is the energy difference between the two levels and g is the coupling constant for the pairing interaction, according to equations (2) and (3) in the following section. A superfluid phase will only occur for a strong enough coupling constant g. This model was invented to test the Bardeen–Cooper–Schrieffer (BCS) theory of super conductivity and is easily solved within its framework, i.e., in terms of Bogoliubov quasi-particles, providing also a very simple and powerful description of pairing forces in nuclei. It is well known that the BCS theory leads to a sharp transition from the normal to the superconducting phase. However, in an actual many-body system with a finite number of degrees of freedom, such as a nucleus, the transition is not sharp but smooth, due to quantum fluctuations which become then very important. Therefore, in order to properly describe the transition region we have to go beyond the mean-field approximation. Well-known boson expansion techniques [1, 6] have been exploited for this purpose by several authors. The idea of describing the dynamics of many-body systems in terms of bosons has a long history. Bosons were introduced in the collective model of Bohr and Mottelson [7] through the quantization of the oscillations of a liquid drop to describe excitations of nuclei. Following the ideas of Sawada [8], the quasi-particle random phase approximation was formulated to describe collective oscillations of spherical nuclei [9]. In this vein, as a natural development, the so-called boson expansion methods were formulated [6, 1]. For the present development, we will consider the Schwinger representation of the $su(2)$ algebra. Following some ideas introduced in a previous paper of Kuriyama *et al* [10], we focus on the two-level pairing model in the framework of the Schwinger boson representation, based on the use of four kinds of boson operators.

The paper is organized as follows. In section 2, the two-level pairing model is presented and the Schwinger boson representation is introduced. In section 3, the model is studied in the framework of the Glauber coherent state. It is observed that the well-known BCS wavefunction is equivalent to the projection of the Glauber coherent state on appropriate eigenspaces of the Casimir operators. Conserving approximations in terms of eigenstates of the constants of motion of the model are studied in section 4. The possibility of describing intrinsically excited states is discussed in section 5.

2. Two-level pairing model

The two-level pairing model describes a system of fermions distributed between two 2Ω-fold degenerate levels. Although, in general, the levels may have different degeneracies, here we focus on the simpler cases in which both levels have the same degeneracy. This model is formulated in terms of creation and destruction operators for fermions, satisfying appropriate anti-commutation relations

$$\begin{aligned}
&\{c_{mi}^\dagger, c_{nj}^\dagger\} = \{d_{mi}^\dagger, d_{nj}^\dagger\} = \{d_{mi}^\dagger, c_{nj}^\dagger\} = \{c_{mi}, c_{nj}\} = \{d_{mi}, d_{nj}\} = \{d_{mi}, c_{nj}\} = 0, \\
&\{c_{mi}^\dagger, c_{nj}\} = \{d_{mi}^\dagger, d_{nj}\} = \delta_{mn}\delta_{ij}, \qquad \{d_{mi}^\dagger, c_{nj}\} = \{d_{mi}, c_{nj}^\dagger\} = 0, \qquad (1)\\
&m, n = 1, \cdots, \Omega, \qquad i, j = 1, 2.
\end{aligned}$$

The building blocks of the Hamiltonian of the model are the generators of the $su(2) \otimes su(2)$ algebra:

$$
\begin{aligned}
N_i &= \sum_{m=1}^{\Omega} \left(c_{mi}^{\dagger} c_{mi} + d_{mi}^{\dagger} d_{mi}\right) - \Omega, \\
A_{i+} &= \sum_{m=1}^{\Omega} c_{mi}^{\dagger} d_{mi}^{\dagger}, \qquad A_{i-} = \sum_{m=1}^{\Omega} d_{mi} c_{mi}, \quad i = 1, 2.
\end{aligned} \tag{2}
$$

The two-level pairing model is defined by the Hamiltonian

$$H = \varepsilon(N_1 - N_2) + g(A_{1+} + A_{2+})(A_{1-} + A_{2-}), \qquad g < 0. \tag{3}$$

This operator admits three constants of motion, namely, the Casimir operators,

$$C_i = \tfrac{1}{4} N_i^2 + \tfrac{1}{2}(A_{i-}A_{i+} + A_{i+}A_{i-}), \qquad i = 1, 2 \tag{4}$$

and the operator

$$N = N_1 + N_2 = 2\Omega - 2\Omega = 0. \tag{5}$$

The zero eigenvalue reflects the assumption of the number of fermions being equal to the degeneracy of each level. The pairing model has been used to test the performance of the BCS wavefunction [4, 5]:

$$|\Phi_{\mathrm{BCS}}\rangle = \exp\left(\sum_{i=1}^{2} \alpha_i \sum_{m=1}^{\Omega} c_{mi}^{\dagger} d_{mi}^{\dagger}\right) |0_F\rangle, \tag{6}$$

where $|0_F\rangle$ is the fermion vacuum. It is well known that the BCS wavefunction works very well everywhere except in the vicinity of the critical point, since the neglected quantum fluctuations play there an important role. It is the main aim of the present article to investigate the possibility of improving the description of that region with the help of the Schwinger representation of the $su(2)$ algebra.

In the Schwinger representation, the $su(2)$ generators are expressed in terms of boson operators. Introducing boson creation and destruction operators such that

$$
\begin{aligned}
&\left[a_i, a_j^{\dagger}\right] = \left[b_i, b_j^{\dagger}\right] = \delta_{ij}, \\
&\left[a_i^{\dagger}, a_j^{\dagger}\right] = \left[b_i^{\dagger}, b_j^{\dagger}\right] = \left[a_i^{\dagger}, b_j^{\dagger}\right] = [a_i, a_j] = [b_i, b_j] = [a_i, b_j] = \left[a_i, b_j^{\dagger}\right] = 0,
\end{aligned} \tag{7}
$$

the generators of the algebra are written

$$\widetilde{N}_i = a_i^{\dagger} a_i - b_i^{\dagger} b_i, \qquad \widetilde{A}_{i+} = a_i^{\dagger} b_i, \qquad \widetilde{A}_{i-} = b_i^{\dagger} a_i, \quad i = 1, 2. \tag{8}$$

In the Schwinger representation, the Hamiltonian reads

$$
\begin{aligned}
H &= \varepsilon\left(a_1^{\dagger}a_1 - b_1^{\dagger}b_1 - a_2^{\dagger}a_2 + b_2^{\dagger}b_2\right) + g\left(a_1^{\dagger}b_1 + a_2^{\dagger}b_2\right)\left(b_1^{\dagger}a_1 + b_2^{\dagger}a_2\right). \\
&= \varepsilon\left(a_1^{\dagger}a_1 - b_1^{\dagger}b_1 - a_2^{\dagger}a_2 + b_2^{\dagger}b_2\right) + g\left(a_1^{\dagger}a_1 + a_2^{\dagger}a_2\right) \\
&\quad + g\left(a_1^{\dagger}b_1^{\dagger}a_1b_1 + a_2^{\dagger}b_2^{\dagger}a_2b_2 + a_2^{\dagger}b_1^{\dagger}a_1b_2 + a_1^{\dagger}b_2^{\dagger}a_2b_1\right),
\end{aligned} \tag{9}
$$

and the constants of motion are written

$$N = a_1^{\dagger}a_1 + a_2^{\dagger}a_2 - b_1^{\dagger}b_1 - b_2^{\dagger}b_2, \tag{10}$$

$$Q_1 = a_1^{\dagger}a_1 + b_1^{\dagger}b_1, \qquad Q_2 = a_2^{\dagger}a_2 + b_2^{\dagger}b_2. \tag{11}$$

The operators Q_1, Q_2 play the same role as the Casimir operators. Under the present conditions, the physical subspace is defined by $N = 0, Q_1 = Q_2 = \Omega$. Therefore, in the physical subspace,

$$a_1^{\dagger}a_1 = b_2^{\dagger}b_2, \qquad b_1^{\dagger}b_1 = a_2^{\dagger}a_2.$$

The boson realization of the pairing model may be safely used to describe the fermion system which it is supposed to represent provided the constants of motion N, Q_1, Q_2 are properly taken into account. In the physical subspace characterized by these constants of motion, the fermion and boson realizations of the model are exactly equivalent.

3. Glauber coherent state

Let us consider the description of our system provided by a Glauber coherent state

$$|\Psi\rangle = \exp\left(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger + \beta_1 b_1^\dagger + \beta_2 b_2^\dagger\right)|0\rangle, \tag{12}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are complex numbers and $|0\rangle$ is the boson vacuum. It may be easily seen that (the boson image of) the BCS wavefunction (6), which will not be considered in the present calculations, is precisely the projection of the Glauber coherent state (12) on the eigenspace of the operators Q_1, Q_2, (equation (11)) corresponding to the common eigenvalue Ω. For the state (12), the expectation value of the Hamiltonian reads

$$\begin{aligned}\mathcal{E} = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} &= \varepsilon(|\alpha_1|^2 - |\beta_1|^2 - |\alpha_2|^2 + |\beta_2|^2) + g(|\alpha_1|^2 + |\alpha_2|^2)\\ &\quad + g(|\alpha_1|^2|\beta_1|^2 + |\alpha_2|^2|\beta_2|^2 + \alpha_2^*\beta_1^*\alpha_1\beta_2 + \alpha_2\beta_1\alpha_1^*\beta_2^*),\\ &= 2\varepsilon(\alpha^2 - \beta^2) + g(\alpha^2 + \beta^2) + 2g\alpha^2\beta^2(1 + \cos(\phi_1 + \psi_2 - \phi_2 - \psi_1)).\end{aligned} \tag{13}$$

The last line follows if we take $|\alpha_1| = |\beta_2| = \alpha, |\beta_1| = |\alpha_2| = \beta, \phi_1 = \arg\alpha_1, \phi_2 = \arg\alpha_2, \psi_1 = \arg\beta_1, \psi_2 = \arg\beta_2$. This assumption is justified recalling that the condition $N = 0$ implies $|\alpha_1|^2 + |\alpha_2|^2 - |\beta_1|^2 - |\beta_2|^2 = 0$ and the conditions $Q_i = \Omega$ imply $|\alpha_i|^2 + |\beta_i|^2 = \Omega, i = 1, 2$, which means that the constants of motion are being implemented in the average. It follows that $|\alpha_1|^2 = |\beta_2|^2, |\alpha_2|^2 = |\beta_1|^2$. For $\phi_1 + \psi_2 - \phi_2 - \psi_1 = 0$, which is the most favourable situation concerning the phases, the expectation value of the Hamiltonian reduces to

$$\mathcal{E} = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} = 2\varepsilon(\alpha^2 - \beta^2) + g\Omega + 4g\alpha^2\beta^2, \qquad \beta^2 = \Omega - \alpha^2, \quad \alpha^2 \leqslant \Omega. \tag{14}$$

The ground-state energy is the minimum of $\mathcal{E}$ and is expressed as

$$\mathcal{E}_0 = -2\varepsilon\Omega + g\Omega, \qquad 0 \geqslant g \geqslant -\frac{\varepsilon}{\Omega}, \qquad \mathcal{E}_0 = \frac{\varepsilon^2 + g^2\Omega(1+\Omega)}{g}, \quad g \leqslant -\frac{\varepsilon}{\Omega}. \tag{15}$$

This quantity, represented in figure 1, reflects the superconducting transition. The critical value of the coupling constant is $g_c = -\varepsilon/\Omega$.

According to its general definition, the Berry phase, needed to describe dynamical processes, reads

$$\mathcal{B} = \mathrm{i}\frac{\langle\Psi|\dot{\Psi}\rangle - \langle\dot{\Psi}|\Psi\rangle}{2\langle\Psi|\Psi\rangle}. \tag{16}$$

For the Glauber coherent state we have

$$\begin{aligned}\mathcal{B} &= \frac{\mathrm{i}}{2}(\alpha_1^*\dot{\alpha}_1 - \dot{\alpha}_1^*\alpha_1 + \alpha_2^*\dot{\alpha}_2 - \dot{\alpha}_2^*\alpha_2 + \beta_1^*\dot{\beta}_1 - \dot{\beta}_1^*\beta_1 + \beta_2^*\dot{\beta}_2 - \dot{\beta}_2^*\beta_2)\\ &= -\alpha^2(\dot{\phi}_1 + \dot{\psi}_2) - \beta^2(\dot{\phi}_2 + \dot{\psi}_1)\\ &= -\frac{1}{2}(\alpha^2 + \beta^2)(\dot{\phi}_1 + \dot{\psi}_2 + \dot{\phi}_2 + \dot{\psi}_1) - \frac{1}{2}(\alpha^2 - \beta^2)(\dot{\phi}_1 + \dot{\psi}_2 - \dot{\phi}_2 - \dot{\psi}_1).\end{aligned}$$

Since $\alpha^2 + \beta^2$ is a constant of motion, the first term is a total derivative and may be dropped from the Lagrangian. For $0 \geqslant g \geqslant -\frac{\varepsilon}{\Omega}$ and for small deviation from equilibrium, the Berry

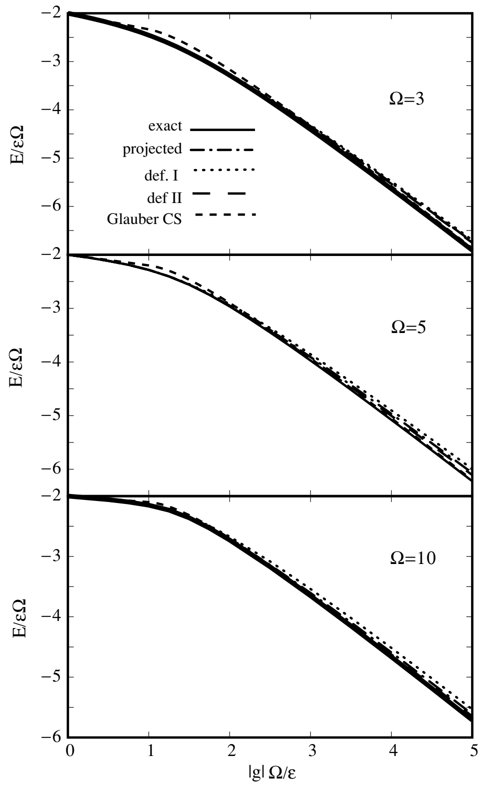

Figure 1. The ground-state energy versus $|g|\Omega/\varepsilon$, for $\Omega = 3$, $\Omega = 5$ and $\Omega = 10$. Variational calculations based on the Glauber coherent state, Glauber coherent state with projection, deformed trial state *I*, deformed trial state *II* and exact result.

phase and the expectation value of the Hamiltonian reduce, respectively, to

$$\mathcal{B} = -\left(\alpha^2 - \frac{\Omega}{2}\right)(\dot{\phi}_1 + \dot{\psi}_2 - \dot{\phi}_2 - \dot{\psi}_1)$$
$$\mathcal{E} = 2\varepsilon(2\alpha^2 - \Omega) + g\Omega + 2g\Omega\alpha^2(1 + \cos(\phi_1 + \psi_2 - \phi_2 - \psi_1)),$$

so that the quadratic Lagrangian becomes

$$\mathcal{L}^{(2)} = \frac{\mathrm{i}}{2}(\gamma^*\dot{\gamma} - \dot{\gamma}^*\gamma) - \mathcal{E}^{(2)}, \qquad \mathcal{E}^{(2)} = 4\varepsilon\gamma^*\gamma + g\Omega(2\gamma^*\gamma + \sqrt{\gamma^*\gamma}(\gamma + \gamma^*)), \tag{17}$$

where the complex number γ is such that $|\gamma| = \alpha$, $\arg\gamma = (\phi_1 + \psi_2 - \phi_2 - \psi_1)$. With the help of the Poisson bracket relations $\{\gamma^*\gamma, \gamma^*\sqrt{\gamma^*\gamma}\}_P = -\mathrm{i}\gamma^*\sqrt{\gamma^*\gamma}$, $\{\gamma^*\gamma, \gamma\sqrt{\gamma^*\gamma}\}_P = \mathrm{i}\gamma\sqrt{\gamma^*\gamma}$, $\{\gamma\sqrt{\gamma^*\gamma}, \gamma^*\sqrt{\gamma^*\gamma}\}_P = -2\mathrm{i}\gamma^*\gamma$, it may be easily seen that this Lagrangian leads trivially to the RPA frequency

$$\omega = 4\varepsilon\sqrt{1 + \frac{g\Omega}{\varepsilon}}. \tag{18}$$

In this connection, it may be observed that the choice of γ is not completely free from ambiguity, since the alternative choice $|\gamma'| = \alpha$, $\arg\gamma' = (\phi_1 + \psi_2 - \phi_2 - \psi_1)/2$ leads to an RPA frequency with half the value in equation (18). This apparently paradoxical result seems to be connected with a long-standing controversy on the RPA frequency associated with the BCS treatment of the pairing model [5]. However, it is clear that while the frequency

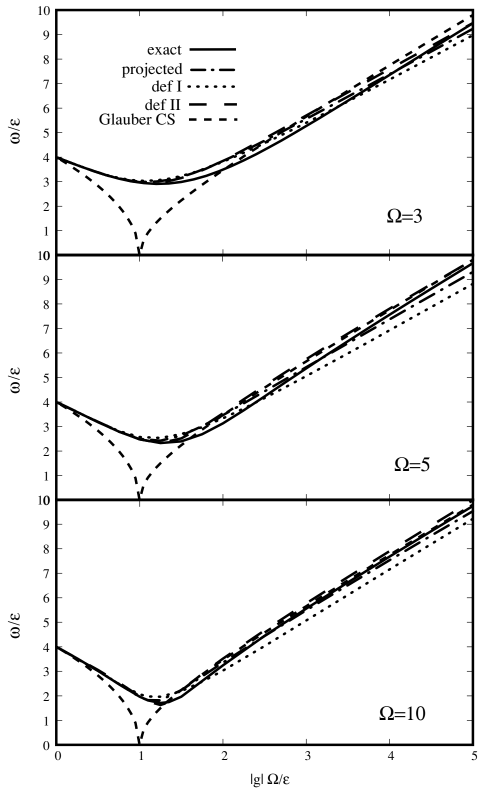

Figure 2. The frequency of small amplitude oscillations, ω, versus $|g|\Omega/\varepsilon$, for $\Omega = 3$, $\Omega = 5$ and $\Omega = 10$. Variational calculations based on the Glauber coherent sate, Glauber coherent state with projection, deformed trial state I, deformed trial state II and exact result.

ω in equation (18) corresponds to excitations within a specific sector characterized by given eigenvalues of the constants of motion Q_1, Q_2, the frequency $\omega/2$ corresponds to certain excitations in which one of these constants of motion is violated, as discussed in section 5, where it is remarked that the model admits also other sectors, characterized by different values of the constants of motion.

For $g \leqslant -\varepsilon/\Omega$, the Lagrangian may be expressed as

$$\mathcal{L} = p\dot{\theta} - \mathcal{E}, \qquad \mathcal{E} = 4\varepsilon p + g\Omega + 2g\left(\frac{\Omega^2}{4} - p^2\right)(1 + \cos\theta), \tag{19}$$

where $\theta = \phi_1 + \psi_2 - \phi_2 - \psi_1$, $p = (\alpha^2 - \beta^2)/2$.

The RPA frequency

$$\omega = 2\sqrt{g^2\Omega^2 - \varepsilon^2}$$

is easily obtained from the corresponding harmonic Lagrangian,

$$\mathcal{L}^{(2)} = \delta p\dot{\theta} + 4g(\delta p)^2 + \frac{g}{4}\left(\Omega^2 - \frac{\varepsilon^2}{g^2}\right)\theta^2. \tag{20}$$

In spite of the fact that the constants of motion are only implemented in the average, it is gratifying that the performance of the Glauber coherent state is rather good. The behaviour of the Glauber RPA frequency, shown in figure 2, is very similar to the behaviour of the

RPA frequency obtained on the basis of the BCS treatment [4, 5] and reflects dramatically the superconducting phase transition. This means, of course, that similarly to the BCS wavefunction, the Glauber coherent state does not take into account quantum fluctuations. Although the RPA frequency is poorly described in the transition region, for large $|g|$ it is very close to the exact excitation energy.

4. Conserving approximations

Since the Glauber coherent state gives a poor description of the system around the transition region, an improvement is required. To this end, we consider eigenstates of the constants of motion which provide what we call, in the following, conserving approximations.

4.1. Glauber coherent state with projection

We begin by considering the trial state vector

$$|\Psi\rangle = \sum_{p=0}^{\Omega} \frac{\alpha^p \beta^{(\Omega-p)}}{(p!(\Omega-p)!)^{3/2}} \left(a_1^\dagger b_2^\dagger\right)^p \left(a_2^\dagger b_1^\dagger\right)^{(\Omega-p)} |0\rangle. \tag{21}$$

It is clear that $N|\Psi\rangle = 0$, $Q_i|\Psi\rangle = \Omega|\Psi\rangle, i = 1, 2$. In subsection 4.2, a certain Glauber coherent state is introduced whose projection on the subspace of good quantum numbers is the image of $|\Psi\rangle$ in equation (21) under Marumori's mapping, this being the motivation for the choice made. We introduce the functions

$$\begin{aligned}
f(x, y, \Omega) &= \frac{(x+y)^{\Omega}}{\Omega!},\\
\gamma(x, y, \Omega) &= x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = (x-y)\frac{(x+y)^{\Omega-1}}{(\Omega-1)!},\\
h(x, y, \Omega) &= xy\frac{\partial^2 f}{\partial x \partial y} = xy\frac{(x+y)^{(\Omega-2)}}{(\Omega-2)!},\\
\eta(x, y, \Omega) &= \sqrt{xy}\sum_{p=1}^{\Omega} \frac{x^{(p-1)} y^{\Omega-p}}{(p-1)!(\Omega-p)!}\sqrt{p(\Omega-p+1)}.
\end{aligned}$$

We find

$$\begin{aligned}
&\langle\Psi|\Psi\rangle = f(\alpha^*\alpha, \beta^*\beta, \Omega),\\
&\langle\Psi|a_1^\dagger a_1 - b_1^\dagger b_1|\Psi\rangle = -\langle\Psi|a_2^\dagger a_2 - b_2^\dagger b_2|\Psi\rangle = \gamma(\alpha^*\alpha, \beta^*\beta, \Omega),\\
&\langle\Psi|a_1^\dagger a_1 + b_1^\dagger b_1|\Psi\rangle = \langle\Psi|a_2^\dagger a_2 + b_2^\dagger b_2|\Psi\rangle = \Omega f(\alpha^*\alpha, \beta^*\beta, \Omega)\\
&\langle\Psi|a_1^\dagger b_1^\dagger a_1 b_1|\Psi\rangle = \langle\Psi|a_2^\dagger b_2^\dagger a_2 b_2|\Psi\rangle = h(\alpha^*\alpha, \beta^*\beta, \Omega),\\
&\langle\Psi|a_1^\dagger b_2^\dagger a_2 b_1|\Psi\rangle = \langle\Psi|a_2^\dagger b_1^\dagger a_1 b_2|\Psi\rangle^* = \frac{\alpha^*\beta}{|\alpha\beta|}\eta(\alpha^*\alpha, \beta^*\beta, \Omega).
\end{aligned}$$

The expectation value of the Hamiltonian may be expressed as

$$\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \mathcal{H} = 2\varepsilon\frac{\gamma(\Omega)}{f(\Omega)} + g\Omega + g\left(2\frac{h(\Omega)}{f(\Omega)} + \frac{(\alpha^*\beta + \alpha\beta^*)}{|\alpha\beta|}\frac{\eta(\Omega)}{f(\Omega)}\right). \tag{22}$$

It may be easily verified that the variational expectation value depends only on the ratio x/y, or α/β, as it should.

The Berry phase reads

$$i\frac{\langle\Psi|\dot\Psi\rangle-\langle\dot\Psi|\Psi\rangle}{2\langle\Psi|\Psi\rangle}=\mathrm{i}\Omega\frac{\alpha^*\dot\alpha-\dot\alpha^*\alpha+\beta^*\dot\beta-\dot\beta^*\beta}{2(\alpha^*\alpha+\beta^*\beta)}=\Omega\frac{\dot\phi+\dot\psi}{2}+\Omega\frac{\alpha^*\alpha-\beta^*\beta}{\alpha^*\alpha+\beta^*\beta}\frac{\dot\phi-\dot\psi}{2},$$

where $\phi=\arg\alpha$, $\psi=\arg\beta$. The term $\Omega(\dot\phi+\dot\psi)/2$ does not contribute to the Lagrangian which finally becomes

$$\mathcal{L}=\frac{\gamma(\Omega)}{f(\Omega)}\frac{\dot\phi-\dot\psi}{2}-2\varepsilon\frac{\gamma(\Omega)}{f(\Omega)}-g\Omega-g\left(2\frac{h(\Omega)}{f(\Omega)}+2\cos(\phi-\psi)\frac{\eta(\Omega)}{f(\Omega)}\right).$$

The frequency ω of small amplitude oscillations is easily determined and is represented in figure 2. This quantity is identified with the excitation energy of the first excited state and is computed as

$$\omega=\sqrt{\frac{AB}{N^2}},$$

where

$$A=\left.\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}s^2}\right|_{s=s_0},\qquad B=-2g\left.\left(\frac{\eta}{f}\right)\right|_{s=s_0}\qquad N=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left.\left(\frac{\gamma}{f}\right)\right|_{s=s_0},\qquad s=\frac{|\alpha|^2}{|\beta|^2},$$

s_0 being the value of s which minimizes $\mathcal{H}$. We conclude observing that equation (21) leads to a very good description of the ground-state and first excited state energies, over the whole range of g values.

4.2. Marumori's method

In this section we apply Marumori's approach [1], to obtain the boson representation of the dynamical variables of our system in terms of the relevant degrees of freedom. The Hilbert space on which the Hamiltonian acts is spanned by the normalized kets

$$|p,q\rangle=\frac{1}{p!q!}\left(a_1^\dagger b_2^\dagger\right)^p\left(a_2^\dagger b_1^\dagger\right)^q|0\rangle.$$

We consider auxiliary bosons c, d, their vacuum $|0)$ and the auxiliary Hilbert space spanned by the normalized kets

$$|p,q)=\frac{1}{\sqrt{p!q!}}c^{\dagger p}d^{\dagger q}|0)$$

and introduce the mapping

$$|p,q\rangle\to|p,q).$$

Under this mapping, the boson images of the operators, $a_1^\dagger b_2^\dagger, a_2^\dagger b_1^\dagger, a_1^\dagger a_1, a_2^\dagger a_2, b_1^\dagger b_1, b_2^\dagger b_2, \ldots,$ are easily obtained. We find

$$a_1^\dagger b_2^\dagger\to\sqrt{(c^\dagger c)}c^\dagger,\qquad a_2^\dagger b_1^\dagger\to\sqrt{(d^\dagger d)}d^\dagger,$$

$$a_1^\dagger a_1\to c^\dagger c,\qquad a_2^\dagger a_2\to d^\dagger d,\qquad b_1^\dagger b_1\to d^\dagger d,\qquad b_2^\dagger b_2\to c^\dagger c.$$

For instance, we find

$$\langle p',q'|a_1^\dagger b_2^\dagger|p,q\rangle=(p',q'|\sqrt{(c^\dagger c)}c^\dagger|p,q)=p\delta_{p'-1,p}\delta_{q',q}.$$

The remaining cases are similarly obtained.

The boson images of the Hamiltonian H and of the trial ket $|\Psi\rangle$ are now easily obtained. We find

$$H \to H_B = 2\varepsilon(c^\dagger c - d^\dagger d) + g(c^\dagger c + d^\dagger d) + g(2c^\dagger d^\dagger cd + \sqrt{(c^\dagger c)}c^\dagger d\sqrt{(d^\dagger d)} + \sqrt{(d^\dagger d)}d^\dagger c\sqrt{(c^\dagger c)}),$$
$$|\Psi\rangle \to |\Psi_B) = \frac{1}{\Omega!}(\alpha c^\dagger + \beta d^\dagger)^\Omega|0).$$

This justifies the choice of $|\Psi\rangle$ in equation (21). In particular, we obtain $\langle\Psi|H|\Psi\rangle = (\Psi_B|H_B|\Psi_B)$, $\langle\Psi|\Psi\rangle = (\Psi_B|\Psi_B)$. The state $|\Psi_B)$ is the projection of the Glauber coherent state associated with the bosons c, d,

$$|\Psi_G) = \exp(\alpha c^\dagger + \beta d^\dagger)|0). \tag{23}$$

4.3. Glauber coherent states with deformation and projection

The state $|\Psi_B)$, except for the normalization factor, may also be written as

$$|\Psi_B) = \exp(Wc^\dagger d)d^{\dagger\Omega}|0).$$

This also suggests considering conserving approximations of the form [11]

$$|\Psi_I) = \exp(W\sqrt{c^\dagger c}c^\dagger d\sqrt{d^\dagger d})d^{\dagger\Omega}|0),$$
$$|\Psi_{II}) = \exp(W\sqrt{d^\dagger d}d^\dagger c\sqrt{c^\dagger c})c^{\dagger\Omega}|0).$$

Except for normalization factors, these states coincide, respectively, with

$$|\Psi_I) = \sum_n c^{\dagger n}d^{\dagger(\Omega-n)}|0)\frac{W^n}{\sqrt{n!((\Omega-n)!)^3}}.$$
$$|\Psi_{II}) = \sum_n c^{\dagger n}d^{\dagger(\Omega-n)}|0)\frac{W^n}{\sqrt{(n!)^3(\Omega-n)!}}.$$

These conserving approximations may be regarded to be based on deformed coherent states.

Then, for the ket $|\Psi_I)$ we find

$$(\Psi_I|\Psi_I) = \sum_{n=0}^{\Omega}\frac{|W|^{2n}}{((\Omega-n)!)^2} = f_I(|W|^2)$$
$$(\Psi_I|c^\dagger c - d^\dagger d|\Psi_I) = \sum_{n=0}^{\Omega}\frac{|W|^{2n}}{((\Omega-n)!)^2}(2n-\Omega) = \gamma_I(|W|^2)$$
$$(\Psi_I|c^\dagger c + d^\dagger d|\Psi_I) = \Omega(\Psi_I|\Psi_I) = \Omega f_I(|W|^2)$$
$$(\Psi_I|c^\dagger cd^\dagger d|\Psi_I) = \sum_{n=0}^{\Omega}\frac{|W|^{2n}}{((\Omega-n)!)^2}n(\Omega-n) = h_I(|W|^2)$$
$$(\Psi_I|\sqrt{c^\dagger c}c^\dagger d\sqrt{d^\dagger d}|\Psi_I) = W^*\sum_{n=1}^{\Omega}\frac{|W|^{2(n-1)}n}{((\Omega-n)!)^2} = \frac{W^*}{|W|}\eta_I(|W|^2).$$

To compute the corresponding Berry phase (16) we need

$$(\Psi_I|\dot{\Psi}_I) - (\dot{\Psi}_I|\Psi_I) = \mathrm{i}\dot{\theta}\sum_n\frac{(2n-\Omega+\Omega)|W|^{2n}}{((\Omega-n)!)^2} = \mathrm{i}\dot{\theta}(\gamma_I(|W|^2) + \Omega f_I(|W|^2)),$$

where $\theta = \arg(W)$. The second term inside the last parenthesis does not contribute to the equations of motion and may be dropped out from the Lagrangian. The expression for the expectation value of the Hamiltonian in state $|\Psi_I)$ is the same as equation (22) with the replacements $f \to f_I, \gamma \to \gamma_I, \eta \to \eta_I, h \to h_I$.

Similarly, for the ket $|\Psi_{II})$ we find

$$(\Psi_{II}|\Psi_{II}) = \sum_{n=0}^{\Omega} \frac{|W|^{2n}}{(n!)^2} = f_{II}(|W|^2)$$

$$(\Psi_{II}|c^\dagger c - d^\dagger d|\Psi_{II}) = \sum_{n=0}^{\Omega} \frac{|W|^{2n}}{(n!)^2}(2n - \Omega) = \gamma_{II}(|W|^2)$$

$$(\Psi_{II}|c^\dagger c + d^\dagger d|\Psi_{II}) = \Omega(\Psi_{II}|\Psi_{II}) = \Omega f_{II}(|W|^2)$$

$$(\Psi_{II}|c^\dagger c d^\dagger d|\Psi_{II}) = \sum_{n=0}^{\Omega} \frac{|W|^{2n}}{(n!)^2} n(\Omega - n) = h_{II}(|W|^2)$$

$$(\Psi_{II}|\sqrt{c^\dagger c}c^\dagger d\sqrt{d^\dagger d}|\Psi_{II}) = W^* \sum_{n=1}^{\Omega} \frac{|W|^{2(n-1)}(\Omega - n + 1)}{((n-1)!)^2} = \frac{W^*}{|W|}\eta_{II}(|W|^2).$$

To compute the corresponding Berry phase we need

$$(\Psi_{II}|\dot{\Psi}_{II}) - (\dot{\Psi}_{II}|\Psi_{II}) = \mathrm{i}\dot{\theta} \sum_n \frac{(2n - \Omega + \Omega)|W|^{2n}}{(n!)^2} = \mathrm{i}\dot{\theta}(\gamma_{II}(|W|^2) + \Omega f_{II}(|W|^2)).$$

The second term inside the last parenthesis does not contribute to the equations of motion and may be dropped out from the Lagrangian. The expression for the expectation value of the Hamiltonian in state $|\Psi_{II})$ is the same as equation (22) with the replacements $f \to f_{II}, \gamma \to \gamma_{II}, \eta \to \eta_{II}, h \to h_{II}$.

In figures 1 and 2, we present, respectively, the ground-state energy and the excitation energy corresponding to the states $|\Psi_I)$ and $|\Psi_{II})$. The performance of $|\Psi_{II})$ is remarkably good.

5. Intrinsic excitations

Our model also admits intrinsic excitations. They occur when some of the single-particle states are blocked up due to asymmetrical occupation. For instance, if r_1 states are asymmetrically filled up in level 1 and r_2 in level 2, with $r_1 + r_2$ even, the number of free fermions becomes effectively $2\Omega - r_1 - r_2$, and the degeneracy of the levels becomes $2\Omega_1 = 2(\Omega - r_1)$ for level 1 and $2\Omega_2 = 2(\Omega - r_2)$ for level 2. The eigenvalue of the operator N remains zero, reflecting the half-filling hypothesis, but $\Omega_1 - \Omega_2 = r_2 - r_1 = 2n$. If $n > 0$, we are led to consider the Hilbert space spanned by the kets

$$|p, q\rangle = \frac{1}{\sqrt{p!(p+n)!q!(q+n)!}} \left(a_1^\dagger b_2^\dagger\right)^p \left(a_2^\dagger b_1^\dagger\right)^q a_1^{\dagger n} b_1^{\dagger n} |0\rangle.$$

This is an invariant subspace of the Hamiltonian. The ket $a_1^{\dagger n} b_1^{\dagger n}|0\rangle$ is the effective vacuum of the sector under consideration. For fixed values of $p + q$, the subspace spanned by these kets cannot be decomposed into invariant subspaces of lower dimension. The methods we have developed throughout may also be used to describe the states belonging to such intrinsically excited sectors. We consider auxiliary bosons c, d, their vacuum $|0)$ and the auxiliary Hilbert space spanned by the kets

$$|p, q) = \frac{1}{\sqrt{p!q!}} c^{\dagger p} d^{\dagger q} |0)$$

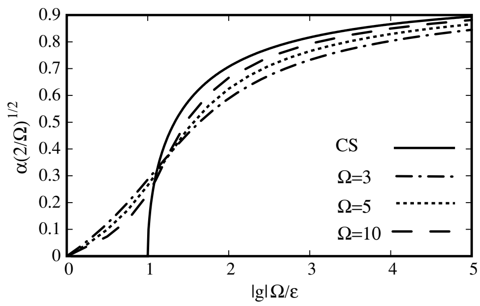

Figure 3. The order parameter. Variational calculations based on the Glauber coherent state and Glauber coherent state with projection for $\Omega = 3$, $\Omega = 5$, $\Omega = 10$.

and introduce the mapping

$$|p, q\rangle \rightarrow |p, q).$$

Under this mapping, the boson images of the operators $a_1^\dagger b_2^\dagger, a_2^\dagger b_1^\dagger, a_1^\dagger a_1, a_2^\dagger a_2, b_1^\dagger b_1, b_2^\dagger b_2, \ldots,$ are easily obtained. We find

$$a_1^\dagger b_2^\dagger \rightarrow \sqrt{c^\dagger c + n}\, c^\dagger, \qquad a_2^\dagger b_1^\dagger \rightarrow \sqrt{d^\dagger d + n}\, d^\dagger,$$
$$a_1^\dagger a_1 \rightarrow c^\dagger c + n, \qquad a_2^\dagger a_2 \rightarrow d^\dagger d, \qquad b_1^\dagger b_1 \rightarrow d^\dagger d + n, \qquad b_2^\dagger b_2 \rightarrow c^\dagger c.$$

If $n < 0$, the effective vacuum should be defined in terms of the operators $a_2^\dagger, b_2^\dagger$, as $a_2^{\dagger|n|} b_2^{\dagger|n|}|0\rangle$. If we wish to go beyond the half-filling hypothesis, the condition $N = 0$ must be relaxed.

Effective Hamiltonians describing intrinsically excited sectors may be obtained by this approach.

6. Order parameter

We wish to comment on the choice of the order parameter whose behaviour illustrates the phase transition associated with our model. It seems natural to choose $\alpha = \sqrt{\langle a_1^\dagger a_1\rangle} = \sqrt{\langle b_2^\dagger b_2\rangle}$, where $\langle X\rangle$ stands for the ground-state expectation value of X. In the framework of the Glauber coherent state we find

$$\alpha = \sqrt{\frac{\Omega}{2} - \frac{\varepsilon}{2|g|}}, \qquad \text{for} \quad \varepsilon < |g|\Omega,$$
$$\alpha = 0, \qquad \text{for} \quad \varepsilon \geqslant |g|\Omega.$$

In figure 3, $\alpha\sqrt{2/\Omega}$ has been plotted as a function of $|g|\Omega/\varepsilon$, for the Glauber coherent state and for the Glauber coherent state with projection, for different values of Ω. The phase transition at $|g| = \varepsilon/\Omega$, is clearly exhibited by the Glauber coherent state. As Ω increases, the results corresponding to the Glauber coherent state with projection approach more and more closely to the Glauber coherent state result.

7. Conclusions

The two-level pairing model was investigated in the framework of the Schwinger representation. It has been shown that zero-point fluctuations, which are important in the crossover region, are well described by certain coherent states which conserve the constants of motion. These so-called conserving approximations provide excellent descriptions of the ground-state energy and of the excitation energy over the whole range of g values, especially the second deformed state, $|\Psi_{II}\rangle$. By excitation energy, we mean the excitation energy to the first excited state belonging to the same sector as the ground-state, namely, the sector characterized by the quantum numbers $\Omega_1 = \Omega, \Omega_2 = \Omega$ and 0 of the constants of motion Q_1, Q_2 and N. The description of the ground-state energy provided by the Glauber coherent state is also reasonable, but not so good. Although the Glauber coherent state fails to provide the description of the excitation energy around the critical point, its performance in the strong coupling limit is very good both for the description of the ground-state energy and of the first excitation energy. It may also be observed that, for $g \to 0$, the excitation energy of the relative ground state of some sectors (for instance, the sector defined by $\Omega_1 = \Omega - 2, \Omega_2 = \Omega, N = 0$) is half the excitation energy within a sector. This fact may explain why, as remarked by some authors [5], the RPA frequency for the pairing model, in the framework of the BCS approach, is half of the correct excitation energy. The present results also show that already at the mean-field level, the bosonic description in the independent particle approximation incorporates important correlation corrections to the conventional mean-field description based on the fermionic realization of the $su(2)$ algebra (BCS theory).

The analysis of the stability of the theory when the initial $su(2)$ symmetry of the model is broken by a small perturbation is an interesting open problem which we hope to consider in a future publication.

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